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Multidimensional basis of p-adic wavelets and representation theory

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Abstract

A multidimensional basis of p-adic wavelets is constructed. The relation of the constructed basis to a system of coherent states (i.e. orbit of action) for some p-adic group of linear transformations is discussed. We show that the set of products of the vectors from the constructed basis and p-roots of one is the orbit of the corresponding p-adic group of linear transformations.

Keywords: p–adic wavelets, multiresolution analysis, representation theory

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1 Introduction

In the present paper we construct a multidimensional p-adic wavelet basis and describe the relation of this basis to representation theory, namely to the theory of coherent states for groups of linear transformations.

The first basis of p-adic wavelets (in one dimension) was constructed in [1]. An analogous basis (with generalizations to some abelian groups) was built in [2]. In [3], [4] some other examples of p-adic wavelet bases were proposed.

An example of multidimensional p-adic wavelet basis for $p=2$ was constructed, with the help of the p-adic multiresolution construction, in [5]. This example coincides (for $p=2$) with the considered in the present paper. We show that this multiresolution wavelet basis can be described by the simple formula (1) below.

In [6] it was shown that the orbit of the action of the p-adic affine group (i.e. the system of coherent states for this group) on a generic function from the space $D_0(\mathbb{Q}_p)$ (of mean zero locally constant compactly supported complex valued functions of p-adic argument) gives a tight uniform frame of wavelets. An explicit parametrization for this frame was obtained and the bound was computed.

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In the simplest case (when we consider the orbit for the wavelet $\psi(x) = \chi(p^{-1}x)\Omega(|x|_p)$, see [7]) the frame of wavelets discussed above contains the products of wavelets from the p-adic wavelet basis and p-roots of one.

In the present paper we construct a basis of multidimensional p-adic wavelets and investigate the relation of this basis with the structure of orbit of p-adic group of transformations. The constructed basis is the direct generalization of the one dimensional p-adic wavelet basis of [1]. In particular wavelets with the support in the unit d-dimensional ball are introduced by the formula

$$\psi_J(x) = \chi(p^{-1}Jx)\Omega(|x|_p),$$

$x \in \mathbb{Q}_p^d$, $|\cdot|_p$ is the d-dimensional p-adic norm, J is the set of representatives from the maximal subballs of a p-adic d-dimensional sphere, Jx is the scalar product in $\mathbb{Q}_p^d$.

We define the multidimensional basis of p-adic wavelets by the formula

$$\psi_{\gamma n J}(x) = p^{-\frac{d\gamma}{2}}\psi_J(p^\gamma x - n), \qquad x \in \mathbb{Q}_p^d, \quad \gamma \in \mathbb{Z}, \quad n \in \mathbb{Q}_p^d/\mathbb{Z}_p^d. \tag{1}$$

This formula is a direct generalization of the construction of the p-adic wavelet basis in one dimension [1].

We show that the frame of wavelets obtained by multiplication of $\psi_{\gamma n J}$ by p-roots of one is the orbit of the group of transformations generated by translations, uniform dilations of all coordinates and by linear transformations which conserve the d-dimensional norm.

This result is the example of the *orbital approach* to p-adic wavelets, proposed in [6]: wavelet frames should be considered as orbits of the corresponding groups of transformations and p-adic wavelet analysis in general is a part of the representation theory for some p-adic groups of transformations.

The exposition of the present paper is as follows.

In Section 2 we recall the earlier constructions of the basis of p-adic wavelets and of the relation between the frames of p-adic wavelets and the theory of coherent states for the affine group.

In Section 3 we introduce the basis of multidimensional p-adic wavelets.

In Section 4 we compare the constructed basis with the multiresolution approach to wavelets.

In Section 5 we show that the introduced multidimensional p-adic wavelet basis can be considered (up to multiplication by p-roots of one) as the orbit of some group of linear transformations and translations.

2 One–dimensional p-adic wavelets

In the present Section we recall the construction of the basis of p-adic wavelets and the relation of the frames of p-adic wavelets with the orbits of the affine group.

Consider a set of wavelets related to the unit ball in the field $\mathbb{Q}_p$ of p-adic numbers in the form of the following complex valued functions of a p-adic argument

$$\psi_k(x) = \chi(p^{-1}kx)\Omega(|x|_p), \qquad x, k \in \mathbb{Q}_p, \tag{2}$$

where $|k|_p = 1$.

Here Ω is the characteristic function of the interval $[0,1]$ (i.e. $\Omega(|x|_p)$ is the characteristic function of the unit ball with the center in zero in $\mathbb{Q}_p$), and χ is the complex valued character of the p-adic argument:

$$\chi(x) = e^{2\pi i\{x\}},$$

where $\{x\}$ is the fractional part of x:

$$\{x\} = \sum_{j=\gamma}^{-1} x_j p^j, \quad x_j = 0, \ldots, p-1$$

for the expansion of the p-adic number x over degrees of p:

$$x = \sum_{j=\gamma}^{\infty} x_j p^j, \quad x_j = 0, \ldots, p-1.$$

We have exactly $p-1$ different functions of the form (2) (considered as functions of x) due to the fact that $\psi_k(x)$ is locally constant as a function of k.

Namely taking the representatives $j = 1, \ldots, p-1$ in the maximal subballs of the sphere $|k|_p = 1$ we get the set of functions

$$\psi_j(x) = \chi(p^{-1}jx)\Omega(|x|_p), \qquad j = 1, \ldots, p-1.$$

All the above functions are dilations of the p-adic wavelet $\psi(x) = \chi(p^{-1}x)\Omega(|x|_p)$:

$$\psi_j(x) = \psi(jx).$$

Therefore the orbit of the group of dilations from the unit sphere acting on the wavelet ψ is exactly the set of wavelets ψ_j.

Then we construct [1] the basis $\{\psi_{\gamma nj}\}$ of wavelets by application to the set of functions $\{\psi_j\}$ of dilations to degrees of p and translations by the representatives of the equivalence classes from the factor group $\mathbb{Q}_p/\mathbb{Z}_p$:

$$\psi_{\gamma nj}(x) = p^{-\frac{\gamma}{2}}\psi_j(p^\gamma x - n), \qquad x \in \mathbb{Q}_p, \quad \gamma \in \mathbb{Z}, \quad n \in \mathbb{Q}_p/\mathbb{Z}_p, \tag{3}$$

$$n = \sum_{i=\beta}^{-1} n_i p^i, \quad n_i = 0, \ldots, p-1. \tag{4}$$

The affine group acts in $L^2(\mathbb{Q}_p)$ by translations and dilations

$$G(a,b)f(x) = |a|_p^{-\frac{1}{2}} f\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{Q}_p, \quad a \neq 0.$$

The orbit of the action of the affine group on the wavelet $\psi(x) = \chi(p^{-1}x)\Omega(|x|_p)$ (i.e. the system of coherent states for this group) will be a frame of wavelets which contains all the products of wavelets from the above basis $\{\psi_{\gamma nj}\}$ of p-adic wavelets and the p different p-roots of one, i.e. $e^{2\pi i p^{-1} l}$, $l = 0, 1, \ldots, p-1$.

In the general case, it was found that [6] for a generic function f (see [6] for the definitions) belonging to the space $D_0(\mathbb{Q}_p)$ of complex valued mean zero locally constant compactly supported

functions of a p-adic argument we have the following theorem: the orbit of this function with respect to action of the affine group is a uniform and tight frame in $L^2(\mathbb{Q}_p)$, and the bound of the frame can be computed explicitly.

Moreover, this orbit possesses the explicit parametrization. The orbit is parameterized by the three indices: the index γ describes the dilations by the degrees of p, the index n describes the translations by elements of $\mathbb{Q}_p/\mathbb{Z}_p$, and the index J takes the finite number of values. Therefore the parameters (translations n and dilations p^γ) which are postulated for the constructions of bases and frames of wavelets arise as parameters on the orbit of a generic function from $D_0(\mathbb{Q}_p)$. Therefore in the p-adic case the multiresolution wavelet analysis is a particular case of the structure of the orbit of the function $f \in D_0(\mathbb{Q}_p)$ with respect to the action of the affine group.

In general, the theory of p-adic wavelets can be considered as a part of the representation theory for p-adic groups of transformations. One can consider orbits of action for the different groups of p-adic linear transformations (i.e. the systems of coherent states for the different groups, see [8]) in the spaces of functions of a multidimensional p-adic argument. In the present paper we investigate two examples of multidimensional p-adic wavelet frames related to systems of coherent states.

3 Multidimensional p-adic wavelet basis

In the present Section we introduce our multidimensional basis of p-adic wavelets. Let us consider (analogously to the one dimensional case) the set of functions

$$\psi_k(x) = \chi(p^{-1}kx)\Omega(|x|_p), \qquad x, k \in \mathbb{Q}_p^d, \tag{5}$$

where $|k|_p = 1$ and

$$kx = \sum_{l=1}^{d} k_l x_l.$$

Let us remind that the d-dimensional p-adic norm is introduced as

$$|x|_p = \max_{l=1,\dots,d} |x_l|_p.$$

Thus the d-dimensional p-adic ball is a direct product of d one dimensional balls, i.e. it coincides with the d-dimensional cube. The d-dimensional sphere is the d-dimensional ball without the subball with the diameter which is p times less than the diameter of the initial ball.

There are $p^d - 1$ different functions $\psi_k(x)$ (as functions of x). We choose the following representatives J for the d-dimensional k, $|k|_p = 1$, which enumerate these functions:

$$k = J = (j_1, \dots, j_d), \qquad j_l = 0, \dots, p-1, \tag{6}$$

where at least one of j_l is not equal to zero. This set of J is the set of representatives from the maximal (d-dimensional) subballs of the p-adic d-dimensional sphere.

We build the basis $\{\psi_{\gamma n J}\}$ of d-dimensional wavelets by application to the set of functions $\{\psi_J\}$ of dilations by the degrees of p and translations by the representatives of the equivalence classes of the factor group $\mathbb{Q}_p^d/\mathbb{Z}_p^d$:

$$\psi_{\gamma n J}(x) = p^{-\frac{d\gamma}{2}} \psi_J(p^\gamma x - n), \qquad x \in \mathbb{Q}_p^d, \quad \gamma \in \mathbb{Z}, \quad n \in \mathbb{Q}_p^d/\mathbb{Z}_p^d, \tag{7}$$

$$n = \left(n^{(1)}, \ldots, n^{(d)}\right), \qquad n^{(l)} = \sum_{i=\beta_l}^{-1} n_i^{(l)} p^i, \quad n_i^{(l)} = 0, \ldots, p-1, \quad \beta_l \in \mathbb{Z}_-. \tag{8}$$

Here $\mathbb{Z}_-$ is the set of negative integers.

Theorem 1 *The set of functions $\{\psi_{\gamma n J}\}$ defined by (7), (8) is an orthonormal basis in $L^2(\mathbb{Q}_p^d)$.*

Proof The proof is analogous to the proof of the corresponding theorem in the one dimensional case [1]. It is easy to see that the wavelet is a mean zero function:

$$\int_{\mathbb{Q}_p^d} \psi_{\gamma n J}(x)\, d\mu(x) = 0. \tag{9}$$

The product of wavelets $\psi_{\gamma n J}\psi_{\gamma' n' J'}$, where $\gamma < \gamma'$, is the wavelet $\psi_{\gamma n J}$ multiplied by a number, since $\psi_{\gamma' n' J'}$ is a constant on the support of $\psi_{\gamma n J}$. This and (9) imply that

$$\langle \psi_{\gamma n J}, \psi_{\gamma' n' J'} \rangle = 0, \qquad \gamma \neq \gamma'.$$

Let us multiply the characteristic functions of the supports of wavelets $\psi_{\gamma n J}$, $\psi_{\gamma' n' J'}$. For $\gamma \leq \gamma'$ the following product is equal to the characteristic function or zero:

$$\Omega\left(|p^\gamma x - n|_p\right)\Omega\left(|p^{\gamma'} x - n'|_p\right) = \Omega\left(|p^\gamma x - n|_p\right)\Omega\left(|p^{\gamma'-\gamma} n - n'|_p\right).$$

For $\gamma = \gamma'$ and n, $n' \in \mathbb{Q}_p^d/\mathbb{Z}_p^d$, we get at the RHS of the above equation

$$\Omega(|n - n'|_p) = \delta_{nn'}.$$

This implies that the scalar product of wavelets $\psi_{\gamma n J}$, $\psi_{\gamma n' J'}$ can be non zero only for $n = n'$.

Computing

$$\langle \psi_{\gamma n J}, \psi_{\gamma' n' J'} \rangle = \delta_{\gamma\gamma'}\delta_{nn'} \int_{\mathbb{Q}_p^d} p^{-\gamma} \chi(p^{\gamma-1}(J' - J)(x - p^{-\gamma} n)) \times$$
$$\times\, \Omega(|p^\gamma x - n|_p)\, d\mu(x) = \delta_{\gamma\gamma'}\delta_{nn'}\delta_{JJ'},$$

we prove the orthonormality of the vectors $\psi_{\gamma n j}$.

To prove that the set of vectors $\{\psi_{\gamma n j}\}$ is an orthonormal basis in $L^2(\mathbb{Q}_p^d)$ (i.e. to prove the completeness of this set) we use the Parseval identity.

Since the set of characteristic functions of balls is complete in $L^2(\mathbb{Q}_p^d)$, and the action of the group of translations $\mathbb{Q}_p^d/\mathbb{Z}_p^d$ and dilations by p^γ, $\gamma \in \mathbb{Z}$ is transitive on the set of balls in $\mathbb{Q}_p^d$ (see lemma 3 below), it is sufficient to prove the Parseval identity for the characteristic function $\Omega(|x|_p)$. We get

$$\begin{gathered} \langle \Omega(|x|_p), \psi_{\gamma n J} \rangle = p^{-\frac{d\gamma}{2}} \theta(\gamma) \delta_{n0}, \\ \theta(\gamma) = 0, \quad \gamma \leq 0, \qquad \theta(\gamma) = 1, \quad \gamma \geq 1. \end{gathered} \tag{10}$$

This implies the Parseval identity for $\Omega(|x|_p)$:

$$\sum_{\gamma n J} |\langle \Omega(|x|_p), \psi_{\gamma n J} \rangle|^2 = \sum_{\gamma=1}^{\infty} (p^d - 1) p^{-d\gamma} = 1 = |\langle \Omega(|x|_p), \Omega(|x|_p) \rangle|^2.$$

This finishes the proof of the theorem. $\square$

4 Comparison with the multiresolution construction

There exists the standard way of constructing of multidimensional wavelet bases from one dimensional bases using the multiresolution construction. In the p-adic case (for $p = 2$) the corresponding example of a multidimensional wavelet basis was considered in [5].

The following definition of the p-adic multiresolution analysis [7], [5], is the direct analogue of the real multiresolution construction, cf. [9], [10].

Definition 2 *A system of closed subspaces $V_\gamma \subset L^2(\mathbb{Q}_p)$, $\gamma \in \mathbb{Z}$, is called a multiresolution analysis (MRA) in $L^2(\mathbb{Q}_p)$ if the following properties are satisfied:*

(i) $V_\gamma \subset V_{\gamma+1}$ for all $\gamma \in \mathbb{Z}$;

(ii) $\bigcup_{\gamma\in\mathbb{Z}} V_\gamma$ is dense in $L^2(\mathbb{Q}_p)$;

(iii) $\bigcap_{\gamma\in\mathbb{Z}} V_\gamma = \{0\}$;

(iv) $f(x) \in V_\gamma \Longleftrightarrow f(p^{-1}x) \in V_{\gamma+1}$ for all $\gamma \in \mathbb{Z}$;

(v) there exists a function $\phi \in V_0$ such that the system $\{\phi(x-n)\}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis in V_0.

Here we as usual take $n \in \mathbb{Q}_p/\mathbb{Z}_p$ in the form of fractions (4). The function ϕ above is called the scaling (refinable) function. The system $\{\phi(p^\gamma x - n)\}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$, will be an orthonormal basis in $V_{-\gamma}$.

Then we define the wavelet spaces W_γ as the orthogonal complements to V_γ in $V_{\gamma+1}$:

$$V_{\gamma+1} = V_\gamma \oplus W_\gamma.$$

We have

$$L^2(\mathbb{Q}_p) = \oplus_{\gamma\in\mathbb{Z}} W_\gamma,$$

(here $\oplus$ is the completion of the direct sum) and for $\gamma \in \mathbb{Z}$

$$f(x) \in W_\gamma \Longleftrightarrow f(p^{-1}x) \in W_{\gamma+1}.$$

Then, taking a final set of functions (wavelets) $\psi_j \in W_0$ such that $\{\psi_j(x-n)\}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis in W_0, we construct the basis of multiresolution wavelets in $L^2(\mathbb{Q}_p)$ of the form $\{p^{-\frac{\gamma}{2}}\psi_j(p^\gamma x - n)\}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$, $\gamma \in \mathbb{Z}$.

Example 1 The p-adic wavelet basis (3) is obtained in the MRA approach by taking

$$\phi(x) = \Omega(|x|_p), \qquad \psi_j(x) = \chi(p^{-1}jx)\Omega(|x|_p), \quad j = 1, \dots, p-1.$$

The multidimensional multiresolution analysis in $L^2(\mathbb{Q}_p^d)$ is introduced by taking the system of tensor product of one dimensional subspaces $V_\gamma^{(l)}$:

$$V_\gamma = \otimes_{l=1}^d V_\gamma^{(l)}, \quad \gamma \in \mathbb{Z}.$$

The scaling function is the tensor product of the one dimensional scaling functions

$$\Phi = \otimes_{l=1}^d \phi^{(l)}.$$

The translations $\{\Phi(p^{\gamma}x-n)\}$, $n\in\mathbb{Q}_p^d/\mathbb{Z}_p^d$, will be an orthonormal basis in $V_{-\gamma}$.

Therefore the system of spaces V_{γ} and the scaling function Φ satisfy the multidimensional generalization of definition 2.

Then we define the wavelet spaces W_{γ} as orthogonal complements to V_{γ} in $V_{\gamma+1}$:

$$V_{\gamma+1}=V_{\gamma}\oplus W_{\gamma},$$

$$L^2(\mathbb{Q}_p^d)=\oplus_{\gamma\in\mathbb{Z}}W_{\gamma}.$$

The multidimensional wavelet spaces have the form of the following direct sums of tensor products of one dimensional wavelet spaces

$$W_{\gamma}=\oplus_{\epsilon_1,\dots,\epsilon_d:\epsilon_k=0,1\backslash\{0,\dots,0\}}\otimes_{l=1}^d V_{\gamma}^{(\epsilon_l)},$$

where $V_{\gamma}^{(0)}=V_{\gamma}$, $V_{\gamma}^{(1)}=W_{\gamma}$, and we take the direct summation of all tensor products of W_{γ} and V_{γ} where not all subspaces are taken to be V_{γ} (i.e. not all indices ϵ are equal to zero).

The multidimensional multiresolution wavelets Ψ_J, $J=(j_1,\dots,j_d)$ are defined by the prescription

$$\Psi_J=\otimes_{l=1}^d\psi_{j_l}, \tag{11}$$

where ψ_j for $j\neq 0$ are equal to one dimensional wavelets ψ_j or equal to ϕ for $j=0$. Here we exclude from the set $\{\psi_J\}$ the product $\otimes_{l=1}^d\phi$ which corresponds to $J=(0,\dots,0)$.

The basis of multiresolution wavelets in $L^2(\mathbb{Q}_p^d)$ is introduced as the set of vectors $\{p^{-\frac{d\gamma}{2}}\Psi_J(p^{\gamma}x-n)\}$, $n\in\mathbb{Q}_p^d/\mathbb{Z}_p^d$, $\gamma\in\mathbb{Z}$ given by translations and dilations of the finite set of the wavelets ψ_J.

Example 2 The p-adic multidimensional wavelet basis (7) is obtained from the above construction in the following way. Since in the dimension one we have

$$\phi(x)=\Omega(|x|_p),\qquad \psi_j=\chi(p^{-1}jx)\Omega(|x|_p),\quad j=1,\dots,p-1,$$

then, taking the tensor product (11) of one dimensional wavelets

$$\Psi_J=\otimes_{l=1}^d\psi_{j_l},\qquad J=(j_1,\dots,j_d)\,,\qquad j_l=0,\dots,p-1,$$

with $J\neq(0,\dots,0)$ we get

$$\Psi_J(x)=\chi(p^{-1}Jx)\Omega(|x|_p),\qquad x\in\mathbb{Q}_p^d,\quad J=(j_1,\dots,j_d)\,,\quad Jx=\sum_{l=1}^d J_l x_l,$$

which coincides with the wavelets (7). Therefore the simple prescription (7) for ψ_J reproduces the more complicated multiresolution construction.

5 Representation theory and frames of wavelets

In the present Section we show the relation of the multidimensional wavelet basis (7), (8) of theorem 1 to the representation theory for some p-adic groups of transformations. For representation theory of p-adic groups see [11].

The proof of the following lemma is straightforward.

Lemma 3 *1) The group $\mathbb{Q}_p^d/\mathbb{Z}_p^d$ of the following lines of fractions with addition modulo one*

$$n = \left(n^{(1)}, \dots, n^{(d)}\right), \qquad n^{(l)} = \sum_{i=\beta_l}^{-1} n_i^{(l)} p^i, \quad n_i^{(l)} = 0, \dots, p-1, \quad \beta_l \in \mathbb{Z}_-,$$

acts transitively by translations on the set of all balls with the diameter one in $\mathbb{Q}_p^d$.

2) The group generated by the above translations from $\mathbb{Q}_p^d/\mathbb{Z}_p^d$ and dilations

$$x \mapsto p^\gamma x, \quad \gamma \in \mathbb{Z}, \quad x \in \mathbb{Q}_p^d,$$

acts transitively on the set of all balls in $\mathbb{Q}_p^d$.

3) A characteristic function of any ball in $\mathbb{Q}_p^d$ can be uniquely represented in the form

$$\Omega(|p^\gamma x - n|), \qquad x \in \mathbb{Q}_p^d, \quad n \in \mathbb{Q}_p^d/\mathbb{Z}_p^d, \quad \gamma \in \mathbb{Z}.$$

Definition 4 *The group O_d is the group of all linear transformations in $\mathbb{Q}_p^d$ which conserve the p-adic norm:*

$$x \mapsto gx, \qquad (gx)_i = \sum_{j=1}^{d} g_{ij} x_j.$$

Here $g \in O_d$, $x \in \mathbb{Q}_p^d$, $|gx|_p = |x|_p$.

This group can be considered as the p-adic analogue of the group of orthogonal transformations in $\mathbb{R}^d$.

Let us denote $\mathcal{V}$ the ball with the diameter one in $\mathbb{Q}_p^d$ with the center in zero. This ball is a $\mathbb{Z}_p$-module.

Lemma 5 *The group O_d is the stabilizer of the unit ball $\mathcal{V}$ in the group of non degenerate linear transformations.*

Proof It is easy to see that O_d is the subgroup of the stabilizer of $\mathcal{V}$.

Let us prove that O_d coincides with the stabilizer. Assume that $g \notin O_d$ belongs to the stabilizer of $\mathcal{V}$. Then there exists $x \in \mathcal{V}$: $gx \in \mathcal{V}$, $|x|_p \neq |gx|_p$. We can assume that $|x|_p < |gx|_p$ (if this will be not satisfied, i.e. we will have $|x|_p > |gx|_p$, then we can consider the element g^{-1} of the stabilizer instead of g and will get $|x|_p < |g^{-1}x|_p$).

Let us normalize x and consider the element $y = x|x|_p$. Then $|y|_p = 1$. We have by the choice of x the inequality $|gy|_p > 1$. Therefore $gy \notin \mathcal{V}$ and g can not belong to the stabilizer of $\mathcal{V}$.

This finishes the proof of the lemma. $\square$

Lemma 6 *The group O_d consists of the following matrices in $\mathbb{Q}_p^d$: a set of columns of a matrix from O_d is a set of vectors of unit norm in $\mathbb{Q}_p^d$ which generates the $\mathbb{Z}_p$-module $\mathcal{V}$.*

The same statement holds for the set of lines of $g \in O_d$.

Proof The n-th column of a matrix of linear mapping is the image of the vector $(0, \dots, 1, \dots, 0)$ with one at the n-th place and zeros at other places. Therefore for the matrix in O_d the norm of

any column is equal to one. For a matrix g the image $g\mathcal{V}$ is the $\mathbb{Z}_p$-module generated by columns of the matrix.

Then the first statement of the lemma follows from the previous lemma.

The module $\mathcal{V}'$ conjugated to $\mathcal{V}$ (i.e. the set of $\mathbb{Z}_p$-linear homomorphisms $\mathcal{V}\to\mathbb{Z}_p$) is isomorphic to $\mathcal{V}$. The elements of $\mathcal{V}'$ can be considered as lines $(k_1,\dots,k_d)$ acting on columns $(x_1,\dots,x_d)$, $k_i,x_i\in\mathbb{Z}_p$ by the $\mathbb{Z}_p$-valued scalar product:

$$kx=\sum_{i=1}^d k_ix_i.$$

The group O_d possesses the natural right action on $\mathcal{V}'$ by application to vectors in $\mathcal{V}'$ of matrices g', $g\in O_d$, where the matrix g' is the transponated matrix to the matrix $g\in O_d$:

$$(g'k)_j=\sum_{i=1}^d g_{ij}k_i.$$

Then, since $g\in O_d$ maps the module $\mathcal{V}$ on itself, the same statement should be satisfied for the conjugated module $\mathcal{V}'$ and the transponated matrix g'. Therefore the lines of the matrix $g\in O_d$ (i.e. the columns of the transponated matrix) should generate the module which coincides with the unit ball with the center in zero.

This finishes the proof of the lemma. $\square$

Remark The above lemma is the p-adic analogue of the proposition which states that the set of columns (lines) of an orthogonal matrix in $\mathbb{R}^d$ is an orthonormal basis in $\mathbb{R}^d$ and, conversely, any matrix with columns (lines) from some orthonormal basis is orthogonal.

Lemma 7 *The group O_d acts transitively on the unit sphere in $\mathbb{Q}_p^d$, i.e. for any pair of vectors from the unit sphere there exists a transformation from O_d which maps one of the vectors to the other.*

Proof Let us choose the basis $\{e_i\}$, $i=1,\dots,d$ in $\mathcal{V}$ with basis vectors $e_i=(0,\dots,1,\dots,0)$ with 1 at the i-th place and zeros at the other places.

It is sufficient to prove the transitivity of O_d for any pair e_i, x, where e_i belongs to the above basis and $|x|_p=1$.

An element of the unit sphere has the form

$$x=(x_1,\dots,x_d)\,,\qquad x_l\in\mathbb{Z}_p,$$

where for at least one of x_l we have $|x_l|_p=1$.

Assume that $|x_1|_p=1$.

Consider the following linear mapping which maps e_1 to x:

$$e_1\mapsto\sum_{l=1}^d x_le_l,\qquad e_i\mapsto e_i,\quad i\neq 1.$$

The matrix of this mapping is obtained from the unit matrix by replacing of the first column by x. Since $\{x,e_2,e_3,\dots,e_d\}$ generate the $\mathbb{Z}_p$-module $\mathcal{V}$, the above linear map lies in O_d.

This finishes the proof of the lemma. $\square$

Remark The same statement is satisfied for the right action of O_d by transponated matrices.

Consider the group of transformations G generated by matrices from O_d, by arbitrary translations, and by the dilations which are homogeneous over all coordinates:

$$x \mapsto p^{\gamma}x, \quad \gamma \in \mathbb{Z}, \quad x \in \mathbb{Q}_p^d.$$

These transformations form three subgroups G_1, G_2, G_3 of G.

We consider the representation of the group G acting in the space $L^2(\mathbb{Q}_p^d)$ by unitary transformations, i.e. matrices in O_d act as

$$f(x) \mapsto f(gx),$$

translations act as

$$f(x) \mapsto f(x+b),$$

and dilations by degrees of p act as

$$f(x) \mapsto p^{-\frac{d\gamma}{2}} f\left(p^{\gamma}x\right).$$

Lemma 8 *An arbitrary element g of the group G can be uniquely represented as a product of elements of the above three subgroups of G, $g^{(1)} \in G_1$, $g^{(2)} \in G_2$, $g^{(3)} \in G_3$:*

$$g = g^{(3)}g^{(2)}g^{(1)}. \tag{12}$$

Here $g^{(1)}$ is a matrix in O_d, $g^{(2)}$ is a translation, and $g^{(3)}$ is a homogeneous dilation $x \mapsto p^{\gamma}x$, $\gamma \in \mathbb{Z}$.

Proof This statement follows from the observation that if we multiply $g \in G$ of the form (12) from the left by the element g' from any of the above three subgroups of G, then the product can be put into the form (12).

The uniqueness of the above representation is straightforward. Assume we have

$$g = g^{(3)}g^{(2)}g^{(1)} = g'^{(3)}g'^{(2)}g'^{(1)}.$$

Then, considering the action of $g = g^{(3)}g^{(2)}g^{(1)} \in G$ in the *affine* space $\mathbb{Q}_p^d$, we see that the length of any vector in the affine space is transformed by the multiplication by $p^{\gamma} = g^{(3)}$. Therefore $g^{(3)}$ in the above expansion is defined uniquely by $g \in G$, i.e. $g^{(3)} = g'^{(3)}$.

Then, considering the equality $g^{(2)}g^{(1)} = g'^{(2)}g'^{(1)}$, and taking into account that the transformation $g^{(2)}g^{(1)}$ maps $0 \in \mathbb{Q}_p^d$ into $g^{(2)}0 \in \mathbb{Q}_p^d$, we get that $g^{(2)}$ is defined uniquely by $g \in G$. This proves the uniqueness of the representation. $\square$

The next theorem gives the interpretation of the d-dimensional basis of wavelets (7), (8) using the system of coherent states (i.e. the orbit of the above representation) for the group G.

Theorem 9 *The orbit of the function $\psi^{(1)}(x) = \chi(p^{-1}x_1)\Omega(|x|_p)$, $x = (x_1, \dots, x_d) \in \mathbb{Q}_p^d$, with respect to the defined above unitary representation of the group G is the frame in $L^2(\mathbb{Q}_p^d)$ which consists of all products of vectors of the p-adic wavelet basis $\{\psi_{\gamma nJ}\}$ given by (7), (8) and p-roots of one.*

Proof Consider the function $\psi^{(1)}(x) = \chi(p^{-1}x_1)\Omega(|x|_p)$. This function is the tensor product of the one dimensional wavelet $\psi(x_1)$ and the characteristic function of the $d-1$-dimensional ball.

One can see that the orbit of the function $\psi^{(1)}$ with respect to the subgroup $G_1 = O_d$ of the d-dimensional linear transformations conserving the d-dimensional norm is exactly the set of wavelets ψ_J, defined by (5), (6). Let us consider

$$\psi^{(1)}\left(g^{(1)}x\right) = \chi\left(p^{-1}\sum_{n=1}^{d} g^{(1)}_{1n}x_n\right)\Omega(|x|_p).$$

Here $g^{(1)}_{mn}$ is the matrix of the linear transformation $g^{(1)} \in O_d$.

The set of numbers $\left(g^{(1)}_{1n}\right)$, as a vector in $\mathbb{Q}_p^d$ lies in the unit sphere by lemma 6.

Thus $\psi^{(1)}\left(g^{(1)}x\right)$ has the form $\psi_k(x)$ for $k \in \mathbb{Q}_p^d$, $k = \left(g^{(1)}_{1n}\right)$, $n = 1, \dots, d$. The local constancy of the function $\psi_k(x)$ with respect to k implies that $\psi^{(1)}\left(g^{(1)}x\right) = \psi_J(x)$ for $J = \left(g^{(1)}_{1n} \bmod p\right)$, $n = 1, \dots, d$. The transitivity of the action of the group O_d on the unit sphere (lemma 7 and the remark after this lemma) implies that the action of $g^{(1)} \in O_d$ on $\psi^{(1)}$ gives all wavelets ψ_J.

An arbitrary translation can be uniquely expanded into the composition of translation of the form (8), translation by integers $0, 1, \dots, p-1$ over any of the coordinates, and translation with the norm less than 1. Translations of ψ_J of the form (8) give all vectors from the basis of wavelets with $\gamma = 0$, translations by the mentioned integers give multiplications by a root of one (or identical transformation), and translations with shorter distances give identical transformations (due to local constancy of the functions under consideration).

Homogenous dilations by p^γ give (7).

This finishes the proof of the theorem. $\square$

Remark An alternative way of introducing a d-dimensional wavelet basis is to take the set of tensor products of d copies of the one dimensional wavelet basis. The corresponding d-dimensional wavelet basis contains the vectors

$$\otimes_{l=1}^{d}\psi_{\gamma_l n_l j_l},$$

where $\psi_{\gamma_l n_l j_l}$ are one dimensional wavelets (3).

This basis possesses the following interpretation as a system of coherent states (i.e. as an orbit of a group action). The orbit of the action of the direct product of the d one dimensional affine groups applied to the tensor product of one dimensional wavelets $\otimes_{l=1}^{d}\psi$ is the frame consisting of the products of vectors from the above basis and p-roots of one.

We see that the different multidimensional wavelet bases (actually the corresponding frames which contains the products of wavelets from the discussed bases and p-roots of one) can be considered as systems of coherent states for the different groups.

This suggests the problem of investigating of the systems of coherent states (orbits in the linear representation, cf. [8]) for the different p-adic groups of linear transformations and of the construction of the corresponding bases and frames of wavelets. This approach should be related to the construction of matrix dilations for multidimensional wavelet bases in the theory of real wavelets (see [10], [9], [12] for a discussion of the matrix dilations).

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